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The Stability of Periodic Orbits

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ABSTRACT

The linear stability analysis of a periodic orbit, $\underline{X}_0(t)$, of a dynamical system $\dot{\underline{X}} = F(\underline{X})$, is related to the anomalous ferromagnetic resonance properties of ferrites. It is shown to have a continuous sequence of normal mode problems associated with it. This sequence defines a natural set of coordinate axes which allow the stability analysis to be put in a form which has already been dealt with analytically in the ferrite resonance theory. A simplification which allowed the basic experimental properties of the ferrites to be accounted for is applied to the stability analysis. The simplified analytic solution is obtained for three-mode systems, its properties and consequences are discussed and it is checked against some rigorous results.

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1. Introduction

Many of the dynamical systems of physics, chemistry and astronomy can be described by an equation of the form

$$\dot{\underline{X}} = \underline{F}(\underline{x}) \tag{1.1}$$

where $\underline{X}$ and $\underline{F}$ are n-dimensional quantities; and $\underline{F}$ is in general a non-linear function of $\underline{X}$, depends on some parameter γ, and has no explicit t-dependence. Examples of such systems are the Rayleigh-Bernard cell,[1,2] and the Couette-Taylor flow system.[3]

The linear stability, or the behaviour of solutions in the immediate vicinity, of a given solution $\underline{X}_0(t)$ of (1.1) is given by the solution of the stability equation:

$$\dot{\underline{x}} = \underline{\underline{\nabla}}(t)\underline{x} \tag{1.2}$$

where

$$\nabla_{ij}(t) = \left.\frac{\partial F_i}{\partial X_j}\right|_{\underline{X}_0(t)} . \tag{1.3}$$

Here $x(t) = \underline{X}(t) - \underline{X}_0(t)$ where $\underline{X}(t)$ is the coordinate vector of a test point moving close to a reference point which is on the given orbit and has coordinate vector $\underline{X}_0(t)$. If $\underline{X}_0(t)$ is a stationary solution, $\underline{\underline{\nabla}}(t)$ is independent of t and solving (1.2) reduces to the diagonalizing of $\underline{\underline{\nabla}}(0)$. If $\underline{X}_0(t)$ is a non-stationary solution, however, $\underline{\underline{\nabla}}(t)$ depends on t. This t-dependence means that the stability analysis cannot be solved by simply diagonalizing $\underline{\underline{\nabla}}$. For $n > 2$ the solution of (1.2) will usually be very complicated and is in general not available in closed form.

Periodic orbits, $\underline{X}_0(t) = \underline{X}_0(t + \frac{2\pi}{\nu})$ are commonly observed experimentally.[1] Some systems exhibit periodic motion at the onset of

turbulence.[4] In the case of periodic orbits, as outlined in Appendix 1, the solution of (1.2) is known to contain important information about the system. Firstly it indicates whether $\underline{X}_0(t)$ is locally stable, and hence whether it represents an experimentally observable state of the system. If it becomes unstable, or bifurcates, a transition in the state of the experimental system is predicted. At a normal bifurcation, a continuous transition occurs and the stability solution predicts whether the new state of the system is periodic or quasiperiodic. It also specifies what the new frequency components appearing in the system's spectrum will be, as well as which will be the fastest growing.

The aim of this paper is two-fold: firstly, to establish a connection between the stability analysis of periodic orbits and the theory of anomalous ferromagnetic resonance in ferrites; and secondly, to exploit this connection to obtain, for three-mode problems, a simplified analytic solution which still contains the essential information concerning the orbit's stability and bifurcations.

Linear differential equations with periodic coefficients, such as (1.2), have been studied at length in their own right.[5] Analytic solutions are, however, generally not available. This paper shows that if the context in which (1.2) arises, namely the stability of a periodic orbit, is considered, rather than treating it mathematically as an isolated equation, an approximate analytic solution of the $n = 3$ case can be obtained.

The stability analysis is shown, in Section 2, to have a continuous sequence of normal modes associated with it. This observation is used, in Section 3, to show that the stability analysis of periodic orbits is in many respects similar to the theory[6] of the anomalous ferromagnetic resonance properties of ferrites. A simplified solution[6] of that problem

was able to account for the basic experimental facts. The sequence of normal modes associated with the stability analysis is used in Section 4 to define a natural set of coordinate axes. Transforming to these axes casts the stability analysis into the same form as the ferrite theory, and the corresponding simplification is made. In section 5 the simplified equations are solved for three mode ($n=3$) systems, and the properties and consequences are discussed. This solution is then checked against some rigorous results in Section 6.

2. A Continuous Sequence of Normal Mode Problems

2.1 Reduction to a Problem of Purely Transverse Motion

The orbit $\underline{X}_0(t)$ is said to be attracting or stable, if the components of $\underline{x}(t) = \underline{X}(t) - \underline{X}_0(t)$ transverse to the orbit shrink to zero as $t \to \infty$. The behaviour, as $t \to \infty$, of the longitudinal component, the component parallel to $\dot{\underline{X}}_0(t)$, of $x(t)$ is not relevant to whether or not $\underline{X}_0(t)$ is stable.

Further, to linear order in $\underline{x}(t)$, the longitudinal component does not appear in the equations of motion for the transverse components. This result is expected intuitively since the linearized theory describes motion arbitrarily close to $\underline{X}_0(t)$. In that region, $\underline{X}_0(t)$ will "look like" a straight line. That is, there will be no characteristic length for changes to occur along the line. In particular, motion transverse to it will be independent of the position along it. A rigorous derivation of this decoupling is given in Appendix 2.

Thus the longitudinal motion is not only less important than the transverse motion, it has, to linear order, no effect on the transverse motion. The linear stability of an orbit in a n-mode system can therefore be reduced to an (n-1)-dimensional problem consisting only of the transverse modes.

It will be seen that the longitudinal motion is the source of the time-dependence of the coefficients in the (n-1)-dimensional transverse problem.

2.2 A Frozen Problem

A problem related to the stability problem will now be considered. The longitudinal motion will be considered "frozen out" of the problem. Pictorially this can be imagined in terms of trapping both the reference and test points between two (n-1)-dimensional glass sheets transverse to $\dot{\underline{X}}_0(t_0)$ and intersecting it at $\underline{X}_0(t_0)$. These sheets are separated by an infinitesimal distance and are "frictionless": they do not affect motion transverse to $\dot{\underline{X}}_0(t)$. Mathematically this frozen problem is achieved (see Appendix 3) by changing to a coordinate frame with one axis, axis n say, parallel to $\dot{\underline{X}}_0(t_0)$ and the others orthogonal to it. Then $\dot{\underline{X}}_n(t)$ is set equal to zero, without changing $\dot{\underline{X}}_T(t)$, where T runs over the transverse directions.

With the longitudinal motion frozen out of the dynamics in this way, the stability problem becomes an analysis of the stability of the fixed point $\underline{X}_0(t_0)$. This intuitively acceptable result is proven in Appendix 3. The coefficients of the stability equations are then constants, and diagonalizing the appropriate matrix reduces the problem to one of (n-1) independent normal modes.

Thus at each point $\underline{X}_0(t_0)$ of the orbit, there is a set of normal modes associated with the stability analysis.

2.3 The Full Stability Analysis

When the longitudinal motion is "unfrozen", and the complete dynamics

considered, the properties of the associated normal modes will be modulated in t, and t-dependent couplings between them will arise. These couplings in general make an exact analytic solution unobtainable. The identification of a sequence of associated normal mode problems means two things, however. Firstly, for every orbit[7] $\underline{X}_0(t)$, of any dynamical system, there is a natural set of coordinate axes in which to discuss stability. Secondly, when this natural set of axes is used, the stability problem of a periodic orbit $\underline{X}_0(t) = \underline{X}_0(t+\frac{2\pi}{\nu})$ is seen to be in a form which has already been dealt with analytically in a different context. This correspondence is displayed in Section III and exploited in Section 4.

3. The Ferrite Analogy

Attention will now be restricted to those cases where $\underline{X}_0(t)$ is periodic: $\underline{X}_0(t) = \underline{X}_0(t + \frac{2\pi}{\nu})$. The stability analysis of periodic orbits will be shown to be similar to the theory of ferromagnetic resonance in ferrites.[6]

Ferrites exhibit two anomalous effects in their microwave absorption properties in ferromagnetic resonance experiments.[8] Firstly the usual resonance is observed to saturate at powers far below those originally thought to be necessary for saturation. Secondly a secondary absorption peak appears at values of the d.c. field below that required for resonance.

Suhl[6] was able to account for the basic experimental facts, analytically and with reasonable precision, by simplifying the equations of motion of the magnetization and solving the simplified equations. He showed that the uniform magnetization, precessing around the d.c. field under the influence of the transverse microwave field, causes a complicated time-dependent, but periodic, coupling between the spin waves in the ferrite. The coupling caused by the precession can, under certain conditions, cause

the spin waves to grow to large non-thermal values. Depending on the conditions, these spin wave instabilities can lead to the two anomalous effects described above.

Like the motion of the transverse components of $\underline{x}(t)$, in the stability analysis of a periodic orbit $\underline{X}_0(t)$, the excitations of a ferromagnet, in the form of spin waves, can be regarded as small amplitude motion and treated to linear order.

If the ferromagnet is a ferrite in the experiment just described, though, the analogy can be taken further. In the absence of the periodic coupling caused by the uniform precession, spin waves can be regarded as normal modes of excitation. If the longitudinal motion is "frozen out" of the stability analysis of $\underline{X}_0(t)$, the problem becomes one of a set of normal modes. The periodic uniform precession in a ferrite produces coupling between spin waves. The periodic longitudinal motion along $\underline{X}_0(t)$ produces a periodic coupling of the normal modes of the frozen problem.

Suhl's analysis shows that, under appropriate conditions, certain spin waves in the ferrite may have exponentially growing amplitudes. When a stable periodic orbit $\underline{X}_0(t)$ bifurcates and becomes unstable, certain components of $\underline{x}(t)$ have exponentially growing amplitudes.

Further, two general classes of bifurcations of periodic orbits allow closer analogies to the ferrite resonance theory. In the first class, the new stable orbit has a fundamental frequency equal to half that of the original orbit. These bifurcations (for which one real eigenvalue of the Poincaré map passes out through the unit circle at -1 : see Appendix 1) are observed[9,10] and are referred to as subharmonic or period-doubling bifurcations. At such a bifurcation components of $\underline{x}(t)$ with a frequency

equal to half that of the original longitudinal motion grow exponentially in amplitude. At the secondary ferrite absorption maximum, it is those spin waves whose frequency is equal to one half that of the uniform precession whose amplitudes grow exponentially.

In a second class of bifurcation of a periodic orbit (those bifurcations for which a real eigenvalue of the linearized Poincare map pass out through the unit circle at +1 : see Appendix 1) components of $\underline{x}(t)$ with a frequency equal to that of the longitudinal motion grow exponentially in amplitude. The premature saturation of the main ferrite absorption peak was shown by Suhl[6] to be due to the growth to large amplitudes of spin waves whose frequency is equal to that of the uniform precession.

Thus, at the levels of both the equations of motion and their solutions, the stability analysis of periodic orbits and the theory of ferromagnetic resonance are very similar problems. Further, it will be seen that, if the simplifying procedure suggested by Suhl is used in appropriate cases for both problems, the equations of motion for the two problems become identical. The analogy[11] is summarized in Table 1 .

Despite the complexity of the equations of motion of the magnetization field in the ferrites, Suhl[6] was able to account for spin-wave instabilities which cause the two anomalous absorption effects and also, in the case of the secondary absorption peak, describe the state of the system beyond the instability threshhold. He used a synchronism argument to simplify the equations of motion. Because of the analogy between the stability analysis of $\underline{X}_0(t)$ and the ferrite resonance theory, and because of the success of Suhl's approach in the latter case, the same synchronism argument will now be used to simplify the stability analysis.

Property	Orbital Stability	Ferrite Resonance
Small amplitude motion	Transverse components of $\underline{x}(t)$	Deviations of local magnetization from from uniform average value
Source of t-dependent, periodic coupling	Longitudinal motion around $\underline{X}_0(t)$ - frequency ν	Precession of uniform magnetization around d.c. field - frequency ν
Normal modes in absence of this coupling	Normal modes of frozen problem	Spin waves
Nature of Instability	Components of $\underline{x}(t)$ with, for example frequency = $\nu/2$ or ν acquire exponentially growing amplitudes	Spin waves with frequency $\nu/2$ (secondary peak) or ν (principal peak) acquire exponentially growing amplitudes

Table 1

4. Simplification of the Stability Analysis

4.1 Suhl's Simplification

The simplification used by Suhl can be described, as follows. The linearized equations of motion (in the ferrite, for the spin waves) and be written as

$$\dot{\underline{y}}(t) = \underline{\underline{M}}(t)y(t) \tag{4.1}$$

where

$$\underline{\underline{M}}(t) = \underline{\underline{M}}(t + \frac{2\pi}{\nu}) \quad .$$

In the absence of the periodic coupling (caused by the uniform precession in the ferrite), $\underline{\underline{M}}$ is diagonalized and the coordinates y represent the uncoupled normal modes of the problem. Suhl assumes that the effect of the coupling is to change the t-dependence of the normal modes only by multiplication by prefactors which vary slowly with respect to the normal mode frequencies.

When the elements of $\underline{\underline{M}}(t)$ are decomposed by complex Fourier transforms, the Fourier components can then be classified as follows. With each coordinate, y_i , in $\underline{y}$ there is associated a normal mode frequency ν_i say. The k'th Fourier component of each element, $M_{ij}(t)$, can be classified according to whether or not $k\nu + \nu_j$ is within $\frac{\nu}{2}$ of ν_i . Those components which do satisfy this synchronism criterion were retained by Suhl. They produce terms in each row of the right-hand side of equation (4.1) with frequency close to that of the term in the same row on the left-hand side. The components which do not satisfy this matching criterion were discarded. Because their frequencies were separated by more than $\nu/2$ from the frequency of the term on the left-hand side, it was claimed that the effect of these terms, particularly over long-time scales, will be small.

A completely equivalent approach is to factor out the time dependence of the normal modes in the absence of the periodic coupling. The coefficients are then assumed to be slowly varying functions of time. The simplification then consists of disregarding all those terms which must be rapidly varying in time.

As indicated earlier, the success of this simplification in accounting for the resonance properties of ferrites, and the analogy between the resonance theory and the stability of periodic orbits, suggests using the same simplification in the latter case. The use of a simplification is also suggested by the fact that all the details of the solution of the stability equation (1.2) are not usually required. The Poincaré map (see App. 1) is obtained by comparing the solution at two values of t separated by the period of $\underline{X}_0(t)$. Thus any oscillations at harmonics of that frequency will have no effect on the Poincaré map. The linking number (see App. 1), roughly the number of times $\underline{X}(t)$ twists around $\underline{X}_0(t)$, is clearly a gross property of the solution: it is possible to perturb the solution by adding many small oscillations and yet not change the linking number at all. Thus any simplification, which only distorts the solution slightly, may leave its most interesting properties essentially unchanged.

The simplification described earlier is most clearly implemented after a coordinate transformation.

4.2 Stability Analysis: A Coordinate Transformation

The coordinate transformation to be performed will be a combination of two transformations. The first, with transformation matrix $\underline{\underline{G}}(t)$, will ensure that one coordinate axis is parallel to the orbit at $\underline{X}_0(t)$, i.e. parallel to $\dot{\underline{X}}_0(t)$. This transformation enables the transverse motion to be decoupled from the longitudinal motion as described in §II. The second

transformation, transforms the axes orthogonal to $\dot{\underline{X}}_0(t)$ so that they coincide with the normal modes of the frozen problem. After this composite transformation has been performed, the transverse components can be considered independently from the longitudinal component, the system is described in terms of associated normal modes and the analogy with the ferrite resonance theory can be exploited.

The details of this procedure are as follows. For simplicity a three-mode system is assumed, but the generalization to $n > 3$ is immediate.

Let $\underline{\underline{G}}(t)$ be any 3×3 matrix whose third column is

$$\underline{C}_3(\underline{\underline{G}}(t)) = \underline{F}(\underline{X}_0(t)) = \dot{\underline{X}}_0(t) \quad . \tag{4.2}$$

The first two columns of $\underline{\underline{G}}$, $\underline{C}_T(\underline{\underline{G}}(t))$, $T = 1,2$ are chosen to be continuous periodic functions of t, and orthogonal to $\underline{C}_3(\underline{\underline{G}}(t))$, but otherwise arbitrary. Let $\underline{\underline{E}}'(t)$ be the 2×2 matrix which diagonalizes $[G^{-1}(t)\underline{\underline{V}}(t)\underline{\underline{G}}(t)]'$. (This prime notation will be used to denote the two dimensional matrix (or vector) formed from the first two rows and columns (or elements) of the primed three dimensional matrix (or vector).) The coordinate transformation to be considered is given by

$$\underline{x}(t) = \underline{\underline{J}}(t)\underline{z}(t) \tag{4.3}$$

where

$$\underline{\underline{J}}(t) = \underline{\underline{G}}(t)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\underline{\underline{E}}'(t)} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \tag{4.4}$$

The stability equation (1.2) then becomes

$$\dot{z}(t) = \underline{\underline{K}}(t)\underline{z} \tag{4.5}$$

where

$$\underline{\underline{K}}(t) = \underline{\underline{J}}^{-1}(t)\underline{\underline{V}}(t)\underline{\underline{J}}(t) - \underline{\underline{J}}^{-1}(t)\dot{\underline{\underline{J}}}(t) \tag{4.6}$$

As is shown in Appendix 2,

$$K_{T3}(t) = 0 \qquad T = 1,2$$

That is the equations of motion of the transverse components of (4.5) are independent of the longitudinal component. Considering only the transverse components then replaces (4.5) by the two-dimensional equation

$$\dot{\underline{z}}'(t) = \underline{\underline{K}}'(t)\underline{z}'(t) \ , \tag{4.7}$$

where $\underline{\underline{K}}'(t) = \underline{\underline{K}}'(t+2\pi/\nu)$.

Further, it is seen from (4.4) that

$$[\underline{\underline{J}}^{-1}\ \underline{\nabla}\underline{\underline{J}}]' = \underline{\underline{E}}'^{-1}[\underline{\underline{G}}^{-1}\ \underline{\nabla}\underline{\underline{G}}]'\underline{\underline{E}}' \tag{4.8}$$

By the definition of $\underline{\underline{E}}'$, this is diagonal. The matrix $[G^{-1}\ \underline{\nabla}\underline{\underline{G}}]'$ is precisely the matrix which appears (see Appendix 3) in the stability analysis of the frozen problem at $\underline{X}_0(t)$. The frequencies of oscillation of the frozen normal modes at $\underline{X}_0(t)$ are therefore given by the imaginary parts of the diagonal elements of $[\underline{\underline{J}}^{-1}\ \underline{\nabla}\underline{\underline{J}}]'$. These frequencies oscillate periodically about their average values. The average values of the frozen normal mode frequencies will be taken as the characteristic normal mode frequencies associated with the stability analysis.

Fourier transforming the terms in $\underline{\underline{K}}'(t)$ gives

$$\begin{aligned} \underline{\underline{K}}'(t) &= \sum_{m=-\infty}^{\infty} e^{im\nu t}\, \underline{\underline{K}}'_m \\ &= \sum_{m=-\infty}^{\infty} e^{im\nu t} \left[\begin{pmatrix} D_{1m} & 0 \\ 0 & D_{2m} \end{pmatrix} + \underline{\underline{C}}'_m \right] \end{aligned} \tag{4.9}$$

where

$$\begin{Bmatrix} D_T \\ \underline{\underline{C}}' \end{Bmatrix}_m = \frac{\nu}{2\pi} \int_0^{2\pi/\nu} \begin{Bmatrix} [\underline{\underline{J}}^{-1} \nabla \underline{\underline{J}}]_{TT} \\ [\underline{\underline{J}}^{-1} \dot{\underline{\underline{J}}}]' \end{Bmatrix} (t) e^{-im\nu t} dt$$

4.3 Simplifying the Stability Analysis

Following the simplification described in §4.1 then means that only those terms $K_m^{TT'}$ in (4.9) which satisfy the following criterion.

$$|\mathrm{Im}(D_{T'0}) - \mathrm{Im}(D_{T0}) + m\nu| \leq \nu/2$$

will be kept. This means that on the diagonal, $\{D_{Tm}, C_m^{TT}: T = 1,2, m \neq 0\}$ will be discarded. All $C_m^{TT'}$, $T \neq T'$ will be discarded, except for $C_{q_{TT'}}^{TT'}$ where $q_{TT'}$ minimizes $|\mathrm{Im}(D_{T'0}) - \mathrm{Im}(D_{T0}) + q_{TT'}\nu|$. Thus (4.7) becomes

$$\dot{z}'(t) = \tilde{\underline{K}}'(t)\underline{z}'(t) \tag{4.10}$$

where

$$\tilde{K}^{TT'} = \begin{cases} D_{T0} + C_0^{TT} & T = T' \\ C_{q_{TT'}}^{TT'} \, e^{iq_{TT'} T} & T \neq T' \end{cases} \tag{4.11}$$

While the case $n = 3$ has been emphasized, the treatment to this point can be applied to any finite n. In the next section, however, only $n = 3$ problems will be considered: (4.10) will be solved for the case of a general three mode (n=3) system.

5. Three Mode (n=3) Problems

5.1 Solution of Simplified Stability Problem

The case of nonlinear, three mode (n=3) dynamical systems will now be considered. Examples of such systems exhibiting periodic orbits are given

in Refs. 9 and 12.

In these cases equations (4.10) and (4.11) give

$$\dot{\underline{z}}' = \begin{pmatrix} D_{10} + C_0^{11} & C_q^{12}\, e^{iq\nu t} \\ C_{-q}^{21}\, e^{-iq\nu t} & D_{20} + C_0^{11} \end{pmatrix} \underline{z}' \tag{5.1}$$

where q is the integer which minimizes

$$|Im(D_{10} - D_{20}) - q\nu| \quad .$$

The equation of motion which was derived and solved by Suhl[7] to describe the behaviour of spin waves in the ferrites is precisely (5.1), with particular values for the parameters: the D's, the C's, and q .

Although the coefficients in (5.1) are t-dependent, the equation can be solved exactly. The following transformation (dropping the primes)

$$\underline{z} = \begin{pmatrix} \exp[(D_{10}+C_0^{11}+a)t] & 0 \\ 0 & \exp[(D_{20}+C_0^{22}-a)t] \end{pmatrix} \underline{z}_1 \tag{5.2}$$

transforms (5.1) into

$$\dot{\underline{z}} = \begin{pmatrix} -a & C_q^{12} \\ C_{-q}^{21} & a \end{pmatrix} \underline{z}_1 \tag{5.3}$$

if

$$a = \frac{1}{2}\,[(D_{20}+C_0^{12}) - (D_{10}+C_0^{11}) + iq\nu]$$

Equation (5.3) has constant coefficients, however, and so is easily solved by diagonalization. The resulting solution of (5.1) is then

$$\underline{z}(t) = \begin{pmatrix} e^{(\bar{\bar{\mu}}+iq\nu/2)t} & \\ 0 & e^{(\bar{\bar{\mu}}-iq\nu/2)t} \end{pmatrix} \underline{\underline{B}} \begin{pmatrix} b_1\, e^{\mu t} \\ b_2\, e^{-\mu t} \end{pmatrix} \tag{5.4}$$

where b_1 and b_2 are arbitrary constants,

$$\underline{\underline{B}} = \begin{pmatrix} C_q^{12} & C_q^{12} \\ \\ -(\bar{\mu}-iq\nu/2) + \mu & -(\bar{\mu}-iq\nu/2) - \mu \end{pmatrix} \tag{5.5}$$

$$\mu = \sqrt{(\bar{\mu}-iq\nu/2)^2 + C_{-q}^{21}\, C_q^{12}} \tag{5.6}$$

$$\bar{\mu} = \frac{1}{2}(D_{10}+C_0^{11}-D_{20}-C_0^{22}) \tag{5.7}$$

and

$$\bar{\bar{\mu}} = \frac{1}{2}(D_{10}+C_0^{11}+D_{20}+C_0^{22}) \tag{5.8}$$

5.2 <u>The Associated Poincaré Map</u>

The linearized Poincaré map is completely determined by the solution of the linear stability problem. It is therefore of interest to determine the eigenvalues of this map implied by the approximate solution (5.4).

The linearized Poincaré map, P(t),(see Appendix 1) is defined by

$$\underline{z}(t + 2\pi/\nu) = \underline{\underline{P}}(t)\underline{z}(t)$$

It is straightforward to show (see Appendix 4) using (5.4), that the eigenvalues η_1,η_2 of $\underline{\underline{P}}(t)$ are independent of t and are given by

$$\eta_{1,2} = (-)^q\, e^{\frac{2\pi}{\nu}(\bar{\bar{\mu}}\pm\mu)} \tag{5.9}$$

As is discussed in Appendix 1, the nature of a normal bifurcation depends strongly on the values taken by these eigenvalues. To see how the different possibilities can arise, it is instructive to consider two

different classes of periodic orbits: those with an associated soft or relaxational frozen problem and those with an associated hard or oscillatory frozen problem.

5.3 Soft and Hard Frozen Problems

The matrix $[\underline{\underline{G}}^{-1}\underline{\underline{V}}\ \underline{\underline{G}}]'$ determining the transverse frozen problem (see 4.2 and Appendix 3) is a real matrix. Its eigenvalues are therefore either real or complex conjugate pairs. Two cases will be considered here.

Case 1: A Soft Frozen Problem

The first case is that of $[\underline{\underline{G}}^{-1}(t)\underline{\underline{V}}(t)\underline{\underline{G}}(t)]'$ having real eigenvalues at every point on $\underline{X}_0(t)$. This means that the normal modes of the frozen problem are purely relaxational and do not sustain oscillations. The frozen problem can therefore be regarded as soft.

Since $[\underline{\underline{G}}^{-1}\underline{\underline{V}}\ \underline{\underline{G}}]'$ is a real 2×2 matrix with real eigenvalues, its eigenvectors are real. Thus $\underline{\underline{E}}'(t)$ and hence $\underline{\underline{J}}(t)$ are real matrices and the Fourier components in (4.8) satisfy

$$D^*_{Tm} = D_{T,-m} \qquad T = 1,2$$

$$\underline{\underline{C}}'^*_m = \underline{\underline{C}}'_{-m}$$

In particular D_{T0}, $T = 1,2$, and $\underline{\underline{C}}'_0$ are real. Therefore the definition after (5.1) gives

$$q = 0$$

From (5.6), (5.7) and (5.8) one then sees that $\bar{\mu}$ and $\bar{\bar{\mu}}$ are real and that μ can be either real or pure imaginary.

Case 2: A Hard Frozen Problem

The second case which will be considered is that of $[\underline{\underline{G}}^{-1}(t)\underline{\underline{V}}(t)\underline{\underline{G}}(t)]'$ having a complex conjugate pair of eigenvalues at every point on $\underline{X}_0(t)$.

This means that the normal mode frequencies have a real part and so the normal modes oscillate. The frozen problem can therefore be described as hard, or oscillatory.

Since $[\underline{\underline{G}}^{-1}(t)\underline{\underline{V}}_F(t)\underline{\underline{G}}(t)]'$ is a real 2×2 matrix with complex conjugate eigenvalues, its eigenvectors also form a complex conjugate pair. It is then seen that the Fourier components in (4.8) satisfy

$$D_{2m} = D^*_{1,-m}$$

$$C^{22}_m = C^{11*}_{-m}$$

$$C^{21}_m = C^{12*}_{-m}$$

Thus q is that integer which minimizes

$$|\mathrm{Im}(D_{10}) - q\nu/2|$$

Further, (5.6) - (5.8) *give that* $\bar{\mu}$ *is imaginary,* $\bar{\bar{\mu}}$ is real, and μ is either real or pure imaginary.

These results for both cases are summarized in Table 2.

5.4 The Properties of the Solution

Several conclusions can now be drawn from the solution (5.4) of the stability analysis.

Firstly, the implications of the expression (5.9) for the eigenvalues, η_1, η_2, of the Poincaré map, will be summarized. When one or both of these eigenvalues passes out through the unit circle in the complex plane, a bifurcation occurs (see Appendix 1). At a normal bifurcation the arguments of the eigenvalues determine whether the new stable orbit is periodic or quasi-periodic. They also provide information about the new generator which appears, in the spectral analysis of the system described

Frozen Problem (Eigenvalues) / Parameters	Soft (Real)	Hard (Complex)
$\bar{\mu}$	$\frac{1}{2}(D_{10}+C_0^{11}-D_{20}-C_0^{22})$ = real	$i\mathrm{Im}(D_{10}+C_0^{11})$ = imaginary
$\bar{\bar{\mu}}$	$\frac{1}{2}(D_{10}+C_0^{11}+D_{20}+C_0^{22})$ = real	$\mathrm{Re}(D_{10}+C_0^{11})$ = real
μ	$+\sqrt{C_0^{12}C_0^{21}+\bar{\mu}^2}$ = real or pure imaginary	$+\sqrt{\lvert C_q^{21}\rvert^2-(\mathrm{Im}(\bar{\mu})-\frac{q\nu}{2})^2}$ = real or pure imaginary
q	0	$q \in \mathbb{Z}: \lvert \mathrm{Im}(D_{10}) - \frac{q\nu}{2} \rvert = \min.$

<u>Table 2</u>

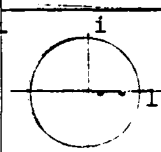
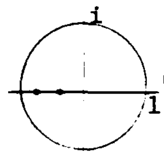
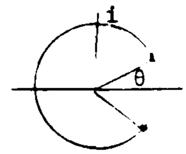
by (1.1), at the bifurcation. The nature of the eigenvalues, as given by (5.9) and these consequences, are summarized in Table 3.

Secondly, some insight into subharmonic or periodic doubling bifurcations can be obtained. Following such a bifurcation, the new orbit is periodic, with a new generator $\nu/2$ in the frequency spectrum. Table 3 indicates that this can only occur when q is odd. If the frozen problem is soft, §5.3 shows that $q = 0$. The present theory thus predicts that subharmonic bifurcations can only occur when the transverse frozen problem, for either all or part of $\underline{X}_0(t)$, is hard, that is sustains oscillations rather than being purely relaxational

When $q = 0$, the oscillating off-diagonal terms in (5.1) can cause instability even when replacing all the coefficients by their average values would indicate stability. This is the phenomenon of parametric resonance,[13] which occurs in many areas of physics and engineering.

A common feature of parametric resonance is the growth of oscillations at a frequency, $\nu/2$, equal to half that of the modulating frequency. In the theory of the secondary absorption peak in ferrites,[6] $q = 1$ and spin waves of frequency half that of the microwave field grow to large amplitudes. Other examples are provided by the original string and tuning fork experiment which led to the introduction of Matthieu's equation,[14] and Faraday crispations[15]: surface wave instabilities also describable by Matthieu's equation.

Thirdly, when the frozen problem is soft and $q = 0$, only average values of the coefficients in the full stability equation are retained. That is, when the transverse frozen problem is purely relaxational, sustaining no oscillations, the frequency-matching argument of § 4.1 indicates that, in the natural set of coordinates defined by the frozen normal modes, the

Property \ μ	Real	Imaginary
Eigenvalues η_1, η_2 of Poincaré Map	 (q even) $\eta_{1,2}$ real  (q odd)	 $\eta_1 = \eta_2^*$, complex
Bifurcation or Instability condition	$\bar{\bar{\mu}} + \mu = 0$	$\bar{\bar{\mu}} = 0$
New Orbit	Periodic	Quasi periodic
New Frequency ν'(mod ν)	ν (q even) $\nu/2$ (q odd)	x <u>or</u> $\nu - x$ where $x = (\mathrm{Im}(\mu) + \frac{q\nu}{2})\ (\mathrm{mod}\ \nu)$

<u>Table 3</u>

The eigenvalues of the Poincaré Map and Their Consequences

(The values of $\bar{\bar{\mu}}, \mu, q$ are given in Table 2)

average values of the coefficients are sufficient to give the basic properties of the solution. This is the reverse case to parametric resonance and can be stated more intuitively as follows. If the normal modes associated with the stability analysis are purely relaxational, and sustain no oscillations, then resonant modulation effects are unlikely to be important.

6. How Reliable is the Simplified Treatment?

The simplification of the stability analysis used in §4 was based on the successful treatment of an analogous problem , the anomalous microwave absorption properties of ferrites. The simplification was not, however, derived in any formal way, for example as part of a perturbation theory approach to the full problem. It is therefore desirable to check the reliability of this simplification. This is done below in several ways using, for example, some simple results from the theory of dynamical systems, the theory of linear differential equations with periodic coefficients, and some exactly soluble cases.

For simplicity most of these checks will be made for the two cases described in § .3: the cases of soft and hard frozen problems.

6.1 Four Basic Properties of the Poincaré Map

There are four basic properties of the Poincaré map which any reasonable approximation scheme should preserve.

Firstly, in a real dynamical system, the Poincaré map must be real. Its eigenvalues must therefore be real or complex conjugate pairs. As indicated in Table 3 this property is preserved by the simplified solution (5.4).

The eigenvalues of the exact linearized Poincaré map $\underline{\underline{P}}(t)$, defined

on a transverse section at $\underline{X}_0(t)$, are independent of t . It is shown in Appendix 4 that the Poincare map resulting from the simplified solution (5.4) preserves this property.

The exact Poincaré map is orientation preserving, that is, the product of its eigenvalues is positive. From (5.9) $\eta_1\eta_2 = e^{4\pi/\nu\ \bar{\bar{\mu}}}$ which is positive since $\bar{\bar{\mu}}$ is real.

Finally, bifurcations of periodic orbits[16] can fall into any of three categories (see Appendix 1). A single real eigenvalue of the Poincaré map may pass out through the unit circle at +1 or at -1 , or a complex conjugate pair of eigenvalues may simultaneously pass out through the unit circle. Table 3 shows that the simplified solution (5.4) preserves this complete set of possible behaviours.

VI.2 Self-Consistency

The simplification described in §4.1 was based on an assumption concerning the nature of the solution of the stability analysis. This assumption can be checked by subtracting from the oscillating (real) frequency of each component of (5.4), the corresponding (real) normal mode frequency, $\mathrm{Im}(D_{T0})$, and seeing whether the difference is small compared to $\nu/2$.

The differences are

$$q\nu/2 \pm \mathrm{Im}(\mu) - \mathrm{Im}(D_{10}) \qquad \text{(1st component)}$$

and

$$-q\nu/2 \pm \mathrm{Im}(\mu) - \mathrm{Im}(D_{20}) \qquad \text{(2nd component)}$$

From the definition of q (§5.1) consistency obtains whenever $\mathrm{Im}(\mu) < \nu/2$. In particular, when μ is real, as is the case for example at subharmonic bifurcations, then the simplification is always self-consistent.

6.3 Diagonal, Exactly Soluble Cases

If the matrix $\underline{\underline{K}}'(t)$ in (4.5) is diagonal, then the simplification described in §4.3 reduces to replacing the coefficients by their average values. It is easy to see, in this case, that the resulting simplified solution always has the exactly correct Poincaré map.

This situation occurs in linear systems, for example, and also in two-mode, $n = 2$, problems. In the latter case, the decoupling described in §2.1 and Appendix 2 reduces the stability analysis to a transverse problem which is one-dimensional and hence exactly soluble. Using either the exact or the simplified solution, one obtains the exact Poincaré map

$$P = \exp\left(\int_0^{2\pi/\nu} \mathrm{Tr}(t)dt\right) \quad ,$$

and regains the Poincaré orbital stability criterion[18]

$$\int_0^{2\pi/\nu} dt\ \mathrm{Tr}(\underline{\underline{V}}(t)) \quad \begin{array}{ll} < 0 & \underline{X}_0(t) \text{ stable} \\ > 0 & \underline{X}_0(t) \text{ unstable} \end{array}$$

6.4 The Liouville-Jacobi Formula

The stability analysis of a periodic orbit is describable by a set of linear differential equations with periodic coefficients. Such equations have been studied extensively, although usually without specific reference to the stability of orbits. One simple result which can be used, however, to test the simplification described in § 4.3, is the Liouville-Jacobi formula.

Consider any set of n linearly independent solutions of the n coupled equations given by

$$\dot{\underline{x}} = \underline{\underline{A}}(t)\underline{x} \tag{6.5}$$

where $\underline{\underline{A}}(t) = \underline{\underline{A}}(2\pi/\nu + t)$. Let $\underline{\underline{X}}(t)$ be the matrix whose columns are these n solutions. The Liouville Jacobi-formula is[18]

$$\det \underline{\underline{X}}(t) = \det \underline{\underline{X}}(0) \quad \exp\left(\int_0^t \mathrm{Tr}\underline{\underline{A}}(t')dt'\right)$$

This formula means that if $\int_0^{2\pi/\nu} \mathrm{Tr}\ \underline{\underline{A}}(t')dt' > 0$, $\det \underline{\underline{X}}(t)$ grows exponentially in t and so (6.5) must have at least one unbounded solution. If $\underline{\underline{A}}(t) = \underline{\underline{K}}'(t)$, as in (4.5) and (6.5) derives from the stability analysis of a periodic orbit, $\underline{X}_0(t)$, then $\underline{X}_0(t)$ must be unstable. The simplification described in §4.3 retains all the $m = 0$ terms on the diagonal of $\underline{\underline{K}}'$. That means that $\int_0^{2\pi/\nu} \mathrm{Tr}\ \underline{\underline{K}}'(t)dt$ is exactly preserved. The rigorous condition given by the Liouville-Jacobi formula is therefore always observed by the simplified solution.

VII. Conclusion

A study of the stability analysis of periodic orbits reveals that this problem is in many respects similar to the theory of ferromagnetic resonance and spin wave instabilities in ferrites.[6]

Exploiting this similarity enables an approximate analytic solution of the general three-mode (n=3) stability analysis to be obtained. Provided $\mathrm{Im}(\mu) < \nu/2$ (see equation (5.6)), as is the case for example whenever subharmonic bifurcations are predicted, the simplification leading to the solution is self-consistent, and satisfies the rigorous checks set out in §6. These are based on some general properties of dynamical systems, the theory of linear differential equations with periodic coefficients, and some exactly soluble cases.

Regarding the stability analysis as an isolated mathematical problem

of coupled differential equations with periodic coefficients, in general does not lead to an analytic solution. However, by exploiting the context of the equations, namely the stability of a periodic orbit, an approximate analytic solution of the $n = 3$ case has been obtained.

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Appendix 1. Mathematical Background for Periodic Orbits

The stability analysis of a periodic orbit, $\underline{X}_0(t)$, which bifurcates often gives information about the "new" orbit, $\underline{X}_1(t)$, which appears after the bifurcation. This appendix indicates when this information is available and lists the mathematical results on which it is based.

A1.1 Two Classes of Bifurcation

Most bifurcations of periodic orbits may be grouped into two classes. Those in the first class will be referred to as "normal" bifurcations. At a normal bifurcation the domain of attraction of the periodic orbit, $\underline{X}_0(t)$, remains of finite size as γ , in (1.1), increases to 0. At $\gamma = 0$, $\underline{X}_0(t)$ becomes unstable and one or more stable orbits, initially coincident with $\underline{X}_0(t)$, start to grow, as γ increases, continuously out from $\underline{X}_0(t)$. The orbit, $\underline{X}_1(t)$, will be related, as described in §§A1.2 and A1.3 , to $\underline{X}_0(t)$. Normal bifurcations are commonly observed experimentally, examples being in the Rayleigh Benard convection cell and the Couette-Taylor flow system.

Bifurcations in the second class will be referred to as "inverted". They are those at which, for $\gamma < 0$, $\underline{X}_0(t)$ has one or more unstable orbits nearby, which, as γ increase to 0 , shrink onto $\underline{X}_0(t)$. The domain of attraction of $\underline{X}_0(t)$ therefore becomes vanishingly small as γ increases to 0 , and hysteretic effects are likely. Orbits in the vicinity of $\underline{X}_0(t)$ are, after the bifurcation, attracted to orbits which bear no particular relation to $\underline{X}_0(t)$.

In summary, at normal bifurcations of periodic orbits, one or more stable orbits appear while at inverted bifurcations one or more unstable orbits disappear.

A linear stability analysis is insufficient to determine whether a bifurcation will be normal or inverted.

A1.2 The Poincaré Map

The Poincaré map is defined by taking a transverse section through a point $\underline{X}_0(t_0)$ on the periodic orbit. Each point $\underline{X}$ on the section can be taken as an initial value for a solution of (1.1). This solution, if the initial point is close enough to $\underline{X}_0(t_0)$, will intersect the section again initially at a point $\underline{Y}$, say. The Poincaré map maps every point $\underline{X}$ on the transverse section onto its "once around image" $\underline{Y}$.

This map can be linearized in the neighbourhood of $\underline{X}_0(t_0)$. The eigenvalues of the linearized Poincaré map, which is real, are either real or in complex conjugate pairs, and are independent of t_0. Since the flow is smooth, the map is orientation preserving and the product of all its eigenvalues is positive. These eigenvalues are also known as Floquet multipliers.

Result 1 The periodic orbit $\underline{X}_0(t)$ is stable if all the eigenvalues of the linearized Poincaré map lie inside the unit circle in the complex plane and becomes unstable and bifurcates when one or more eigenvalues pass outside the unit circle.

In almost all cases[16] either one real or two complex conjugate eigenvalues pass through the unit circle at $\gamma = 0$.

Results 2 to 5 apply only to normal bifurcations. They refer to quantities whose values depend on γ, and the results apply to the limiting values as $\gamma \to 0$.

The periodic orbit $\underline{X}_0(t)$ will have a Fourier spectrum with components at a fundamental, ν, and its harmonics. At a normal bifurcation, a "new" orbit $\underline{X}_1(t)$ will have Fourier components which can be generated by a set of two generators, $\{\nu,\nu'\}$. This condition defines ν' up to

multiples of ν . Use will be made of the modulo notation, defined to satisfy $0 < a(\text{mod } b) \leq b$.

Result 2 If, at a normal bifurcation,

a) one real eigenvalue of the linearized Poincaré map passes through the unit circle, $X_1(t)$ is periodic. The frequency ν' (mod ν) is the new fundamental and is ν if the eigenvalue is positive and $\nu/2$ if the eigenvalue is negative. [For $\underline{X}_1(t)$ periodic, the new Fourier components can be generated by ν' (mod ν) alone.]

b) two complex conjugate eigenvalues of the linearized Poincaré map pass through the unit circle at arguments $\theta, 2\pi-\theta (0 < \theta < 2\pi)$, $\underline{X}_1(t)$ is quasi-periodic and

$$\nu'(\text{mod } \nu) = \text{either } (\theta/2\pi)\,\nu \text{ , or } [(2\pi-\theta)/2\pi]\nu$$

(The proof of this result only works when the eigenvalues η satisfy $\eta^3 \neq 1 \neq \eta^4$)

Corollary of Result 2

Whenever a periodic orbit, of fundamental frequency ν , bifurcates to an orbit with no sharp Fourier component at ν , for example a strange attractor, the bifurcation is inverted and hysteresis effects are likely.

A1.3 The Linking Number

The linking number of a curve close to $\underline{X}_0(t)$ can be regarded, when $n = 3$, as the (real) number of times the curve twists around $\underline{X}_0(t)$, between leaving a transversal of $\underline{X}_0(t)$ and first returning to it. For $n > 3$ a linking number can be defined in terms of an appropriate three-dimensional neighbourhood of $\underline{X}_0(t)$.

As stated in §1.2 , the results listed here refer to values of the relevant quantities as $\gamma \to 0$ at a normal bifurcation.

Result 3

The linking number, ℓ , of $\underline{X}_1(t)$ is the same as the linking number of an orbit $\underline{X}(t)$ arbitrarily close to $\underline{X}_0(t)$.

For such an orbit, $\underline{X}(t)$, the equation of motion for $\underline{X}(t) - \underline{X}_0(t)$ may be taken as linear, and is in fact the stability equation (see §2). The solution of the stability analysis therefore determines the linking number of $\underline{X}(t)$ and hence of $\underline{X}_1(t)$, the new stable solution.

Result 4

$$\nu'(\text{mod } \nu) = \ell\nu(\text{mod } \nu)$$

In the case of a normal bifurcation to a quasi-periodic orbit, determining ℓ thus determines which of the two possibilities in Result 2b, $(\theta/2\pi)\nu$ or $[(2\pi-\theta)/2\pi]\nu$, is correct.

The next result has not been proven but seems a reasonable conjecture.

Result 5

Of the Fourier component amplitudes of $\underline{X}_1(t)$ at frequencies $\nu' + m\nu$ for integral m , the fastest growing one is $\ell\nu$.

(If $\nu' = \nu$, the Fourier components $\nu' + m\nu$ already have a nonzero amplitude at the bifurcation, so "fastest growing" can be taken to mean having the largest discontinuity in the derivative with respect to γ .)

Appendix 2. Decoupling the Longitudinal and Transverse Motion

For an n-mode system, the coordinate transformation set out in §4.2 ensures that the n'th coordinate axis is always parallel to the orbit at $\underline{X}_0(t)$, and the other axes are always orthogonal to it. It will be shown here that $K_{Tn} = 0$ for $T = 1,\ldots,n-1$ where $\underline{\underline{K}}(t)$ is given by (4.5). Equation (4.4) then indicates that, to linear order , the motion of the transverse components is completely unaffected by the longitudinal component, and can be considered separately.

From (4.5)

$$K_{Tn}(t) = \sum_j J^{-1}_{Tj} \left((\underline{\underline{\nabla}}\underline{\underline{J}})_{jn} - \dot{J}_{jn} \right) \tag{A2.1}$$

Using (4.2) and (4.4) gives

$$J_{jn}(t) = F_j(\underline{X}_0(t)) \tag{A2.2}$$

Thus

$$\dot{J}_{jn} = \sum_k \left.\frac{\partial F_j}{\partial X_k}\right|_{X_0(t)} \dot{X}_{0k}(t)$$

$$= \sum_k \nabla_{jk} \quad F_k(\underline{X}_0(t))$$

$$= (\underline{\underline{\nabla}}\ \underline{\underline{J}})_{jn}$$

using (A2.2). Substituting this into (A2.1) gives

$$K_{Tn}(t) = 0 \qquad T = 1,2,\ldots,(n-1) ,$$

as required

(This decoupling clearly occurs whether or not $\underline{X}_0(t)$ is periodic.)

Appendix 3. The Frozen Problem

The purpose of this appendix is to make the notion of the frozen problem, described qualitatively in §2.2, precise and to prove that this "freezing out" of the longitudinal motion reduces the stability analysis to that of a fixed point.

The first step in defining the frozen problem is transforming to a fixed set of axes, one of which is orthogonal to all the others and parallel to the periodic orbit at one point on it, $\underline{X}_0(t_0)$ say. The matrix $\underline{\underline{G}}(t_0)$, where $\underline{\underline{G}}(t)$ is defined in §4.2, achieves this transformation. If $\underline{X} = \underline{\underline{G}}(t_0)\underline{U}$ then (1.1) becomes (dropping t_0 from the notation)

$$\dot{\underline{U}}(t) = \underline{\underline{G}}^{-1}\underline{F}(\underline{\underline{G}}\,\underline{U}(t)) \tag{A3.1}$$

The frozen probelm is then obtained by setting the longitudinal velocity equal to zero. That is, the frozen problem is

$$\dot{V}_T(t) = [\underline{\underline{G}}^{-1}\underline{F}(\underline{\underline{G}}\,\underline{V}(t))]_T \qquad T = 1,2,\ldots,n-1 \tag{A3.2}$$

$$\dot{V}_n(t) = 0 \tag{A3.3}$$

The orbits $\underline{V}(t)$ and $\underline{V}_0(t)$ corresponding to $\underline{X}(t)$ and $\underline{X}_0(t)$ in the full stability problem are specified by the initial conditions

$$\begin{aligned} \underline{V}_0(0) &= \underline{U}_0 \\ \underline{V}(0) &= \underline{U}_0 + \underline{v}(0) \end{aligned} \tag{A3.4}$$

where

$$\underline{\underline{G}}\,\underline{U}_0 = \underline{X}_0(t_0) \tag{A3.5}$$

and

$$v_n(0) = 0 \tag{A3.6}$$

This last condition means that the test particle, as well as the reference particle, is at $t = 0$ on the hyperplane transverse to the orbit, i.e. is between the two "glass sheets" of §2.2

If $\underline{v}(t) = \underline{V}(t) - \underline{V}_0(t)$, then to linear order in $\underline{v}$, (A3.2) and (A3.3) give

$$\dot{v}_T = [G^{-1}\underline{\underline{\nabla}}_F(t)\underline{\underline{G}}\ \underline{v}]_T \qquad T = 1,\dots, n-1 \tag{A3.7}$$

$$\dot{v}_n = 0 \tag{A3.8}$$

where

$$\nabla_{F_{ij}}(t) = \left.\frac{\partial F_i}{\partial X_j}\right|_{\underline{\underline{G}}\ \underline{V}_0(t)} \tag{A3.9}$$

Equations A3.7 to A3.9 constitute the frozen stability problem.

From (A3.6) and (A3.8) , $v_n(t) = 0$ for all t . This means that (A3.7) can be written purely in terms of the (n-1) transverse components.

$$\dot{v}' = [\underline{\underline{G}}^{-1}\underline{\underline{\nabla}}_F(t)\underline{\underline{G}}]'\underline{v}' \tag{A3.10}$$

It can also be seen that $\underline{V}_0(t) = \underline{V}_0(0)$ for all t. From (A3.2),

$$\begin{aligned}\dot{V}_{0T}(0) &= [\underline{\underline{G}}^{-1}\underline{F}(\underline{\underline{G}}\ \underline{V}_0(0))]_T \\ &= [\underline{\underline{G}}^{-1}\underline{F}(\underline{X}_0(t_0))]_T \\ &= 0 \qquad \text{for } T = 1,2,\dots,n-1\end{aligned}$$

since $\lambda\underline{F}(\underline{X}_0(t_0))$ is the n-th column of $\underline{\underline{G}}$ (§4.2). Combining this result with (A3.3) gives that $\dot{\underline{V}}_0(0) = 0$ and so $\underline{V}_0(t) = \underline{V}_0(0)$ for all t .

This means, though, that $\underline{\underline{\nabla}}_F(t)$ (see (A3.9)) is independent of t . That is the frozen stability problem (A3.10) is the stability analysis of a

fixed point, $\underline{V}_0(t) = \underline{V}_0(0)$, and so the coefficients , $\underline{\underline{V}}_F(t) = \underline{\underline{V}}_F(0)$, are constants.

Appendix 4. The Eigenvalues of $P(t)$

The purpose of this Appendix are firstly to show that the simplified solution (5.4) has a Poincaré map $\underline{\underline{P}}(t)$ whose eigenvalues are independent of t , and secondly to determine these eigenvalues.

For the exact solution an operator $\underline{\underline{T}}(t,t')$ can be defined such that

$$\underline{z}'(t') = \underline{\underline{T}}(t,t')\underline{z}'(t) \tag{A4.1}$$

It satisfies

$$\underline{\underline{T}}(t + \frac{2\pi}{\nu} , t' + \frac{2\pi}{\nu}) = \underline{\underline{T}}(t,t') \tag{A4.2}$$

These two results together mean that

$$\underline{\underline{P}}(t) = \underline{\underline{T}}^{-1}(t,t')\underline{\underline{P}}(t')\underline{\underline{T}}(t,t')$$

so that the eigenvalues of the exact Poincaré map, $\underline{\underline{P}}(t)$ are independent of t .

If $\underline{z}'(t)$ is the simplified solution, the same result is established by finding an operator $T(t,t')$ which satisfies (A4.1) and (A4.2) for that solution. Such an operator can easily be constructed using (5.4) . It is

$$\underline{\underline{T}}(t,t') = \begin{pmatrix} e^{(\bar{\bar{\mu}}+ \frac{iq\nu}{2})t'} & 0 \\ 0 & e^{(\bar{\bar{\mu}}- \frac{iq\nu}{2})t'} \end{pmatrix} \underline{\underline{B}} \begin{pmatrix} e^{\mu(t'-t)} & 0 \\ 0 & e^{-\mu(t'-t)} \end{pmatrix} \underline{\underline{B}}^{-1} \begin{pmatrix} e^{-(\bar{\bar{\mu}}+ \frac{iq\nu}{2})t} & 0 \\ 0 & e^{-(\bar{\bar{\mu}}- \frac{iq\nu}{2})t} \end{pmatrix} \tag{A4.3}$$

which is easily seen to satisfy (A4.1) and (A4.2)

Hence the eigenvalues of $P(t)$, as determined from the simplified solution, are independent of t.

From (A4.1) ,

$$\underline{\underline{P}}(t) = \underline{\underline{T}}(t, t + \frac{2\pi}{\nu})$$

Putting $t = 0$ and using (A4.3) then shows that the eigenvalues of the Poincaré map associated with the simplified solution are

$$\eta_{1,2} = (-)^q \; e^{\frac{2\pi}{\nu}(\bar{\bar{\mu}} \pm \mu)}$$

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The linear stability analysis of a periodic orbit, $X_0(t)$, of a dynamical system $\dot{X} = F(X)$, is related to the anomalous ferromagnetic resonance properties of ferrites. It is shown to have a continuous sequence of normal mode problems associated with it. This sequence defines a natural set of coordinate axes which allow the stability analysis to be put in a form which has already been dealt with analytically in the ferrite resonance theory. A simplification which allowed the basic experimental properties to be accounted for is aplied to the stability analysis. The simplified analytic solution is obtained for three-mode systems, its properties and consequences are discussed and it is checked against some rigorous results.

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